Int. J. Heat Mass Transfer. Vol. 30, No. 11, pp. 2441–2444, 1987
Printed in Great Britain

0017-9310/87 \$3.00 + 0.00
Pergamon Journals Ltd.

TECHNICAL NOTES

A complete analytical solution to laminar heat transfer in axisymmetric stagnation point flow

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(*Received* 18 *March* 1986 *and in final form* 6 *November* 1986)

1. INTRODUCTION

THE OBJECTIVE of the present study is to apply Gyarmati's variational principle, which offers a genuine treatment of thermodynamics of irreversible processes, to obtain a rapid analytical solution to laminar flow and heat transfer in axisymmetric stagnation point boundary layer flow. According to boundary layer theory the irreversible processes of momentum and heat transfer in flows around bodies occur mainly inside a thin layer adjacent to the surface of the body. Therefore, it is quite appropriate to study these non-equilibrium processes by the method of irreversible thermodynamics. The variational principle is formulated for the present system and the non-linear boundary layer equations are reduced to simple polynomial equations.

2. BASIC EQUATIONS

The viscous and thermal boundary layer equations for steady, incompressible, laminar and axisymmetric stagnation point flow are

$$\frac{\partial(XU)}{\partial X}+\frac{\partial(XV)}{\partial Y}=0 \quad \text{(continuity)}$$

$$U\frac{\partial U}{\partial X}+V\frac{\partial U}{\partial Y}=U_\infty\frac{\mathrm{d}U_\infty}{\mathrm{d}X}+\gamma\frac{\partial^2 U}{\partial Y^2} \quad \text{(momentum)}$$

and

$$U\frac{\partial T}{\partial X}+V\frac{\partial T}{\partial Y}=\alpha\frac{\partial^2 T}{\partial Y^2} \quad \text{(energy)} \tag{1}$$

where $U_\infty(X)=aX$. The associated boundary conditions are

$$\begin{aligned} Y=0: &\quad U=V=0, \quad T=T_0(X) \\ Y=\infty: &\quad U=U_\infty(X), \quad T=T_\infty \end{aligned} \tag{2}$$

where T_0 and T_∞ satisfy the power law

$$T_0-T_\infty=cX^n. \tag{3}$$

With the help of Mangler's transformation

$$x=X^3/3l^2 \quad \text{and} \quad y=XY/l \tag{4}$$

the conservation equations (1) are transformed into the following equations:

$$\begin{aligned} &\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}=0 \\ &u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}=U_\infty\frac{\mathrm{d}U_\infty}{\mathrm{d}x}+\gamma\frac{\partial^2 u}{\partial y^2} \\ &u\frac{\partial T}{\partial x}+v\frac{\partial T}{\partial y}=\alpha\frac{\partial^2 T}{\partial y^2}. \end{aligned} \tag{5}$$

Since

$$U_\infty(x)=a(3l^2x)^{1/3}=bx^{1/3} \tag{6}$$

the conservation equations (5) correspond to the two-dimensional flow over a wedge of angle $\pi/2$.

3. FORMULATION OF GYARMATI'S PRINCIPLE

The variational principle developed by Gyarmati on the basis of thermodynamic principles which is well known as 'The Governing Principle of Dissipative Processes' is given in its energy picture [1, 2] as

$$\delta\int_V(T\sigma-\psi^*-\phi^*)\,\mathrm{d}V=0 \tag{7}$$

where $T\sigma$ is the energy dissipation.

The variational principle (7) for the present problem assumes the following form:

$$\delta\int_0^l\int_0^\infty\left[-J_q\frac{\partial\ln T}{\partial y}-P_{12}\frac{\partial u}{\partial y}-\frac{L_\lambda}{2}\left(\frac{\partial\ln T}{\partial y}\right)^2-\frac{L_s}{2}\left(\frac{\partial u}{\partial y}\right)^2-\frac{J_q^2}{2L_\lambda}-\frac{P_{12}^2}{2L_s}\right]\mathrm{d}y\,\mathrm{d}x=0. \tag{8}$$

The essence of the present variational procedure is the reformulation of the original set of partial differential equations (5) in terms of the variational principle (8).

4. SOLUTION PROCEDURE

Let us assume that the velocity and temperature distributions in their respective boundary layers are the following polynomials:

$$\begin{aligned} &u/U_\infty=3y/d_1-3y^2/d_1^2+y^3/d_1^3 \quad (y<d_1); \quad u=U_\infty \quad (y\geqslant d_1) \\ &(T-T_\infty)/(T_0-T_\infty)=1-3y/2d_2+y^3/2d_2^3 \quad (y<d_2); \quad T=T_\infty \quad (y\geqslant d_2). \end{aligned} \tag{9}$$

These profiles satisfy the following conditions:

$$\begin{aligned} y=0:&\quad u=0, \quad T=T_0(x), \quad \frac{\partial^2 T}{\partial y^2}=0 \\ y=d_1:&\quad u=U_\infty(x), \quad \frac{\partial u}{\partial y}=0, \quad \frac{\partial^2 u}{\partial y^2}=0 \\ y=d_2:&\quad T=T_\infty, \quad \frac{\partial T}{\partial y}=0. \end{aligned} \tag{10}$$

The smooth-fit conditions $\partial u/\partial y=0$ and $\partial T/\partial y=0$ correspond to $P_{12}=0$ and $J_q=0$ at the respective edges of the boundary layers. Both d_1 and d_2 are unknown parameters and they are to be determined by the variational process.

The velocity and temperature functions (9) are substituted in the momentum and energy balance equations (5) and on integration with respect to y with the help of smooth-fit

NOMENCLATURE

d_1	hydrodynamical boundary layer thickness
d_2	thermal boundary layer thickness
J_q	heat flux
L_1, L_2	Lagrangian densities
L_λ, L_s	conductivities
P_{12}	momentum flux
V	volume of continuum.

Greek symbols

δ	symbol for variation
Δ	ratio of d_2 to d_1
σ	entropy production
ψ^*, ϕ^*	local dissipation potentials in energy picture.

boundary conditions, the fluxes P_{12} and J_q are obtained, respectively. The expression for P_{12} remains the same for any Prandtl number P. But the energy flux J_q assumes different expressions for $P \leqslant 1$ and $P \geqslant 1$, respectively. When $P \leqslant 1$, the expression for J_q in the range $d_1 \leqslant y \leqslant d_2$ is obtained first and the expression for J_q in the range $0 \leqslant y \leqslant d_1$ is determined subsequently by matching the J_q expressions of the two regions at the interface.

Using the expressions of P_{12} and J_q along with the velocity and temperature functions (9) the variational principle (8) is formulated. After carrying out the integration with respect to y one can obtain the variational principle in the following form:

$$\delta \int_0^l L_1(d_1, d_2, d_1', d_2')\,\mathrm{d}x = 0 \quad (P \leqslant 1) \tag{11}$$

and

$$\delta \int_0^l L_2(d_1, d_2, d_1', d_2')\,\mathrm{d}x = 0 \quad (P \geqslant 1) \tag{12}$$

where primes indicate differentiation with respect to x. The variational principles (11) and (12) are found identical in the case $d_1 = d_2$. The parameters to be varied independently in equations (11) and (12) are d_1 and d_2. Accordingly, the Euler–Lagrange equations are

$$\frac{\partial L_{1,2}}{\partial d_1} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial L_{1,2}}{\partial d_1'}\right] = 0 \tag{13}$$

and

$$\frac{\partial L_{1,2}}{\partial d_2} - \frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{\partial L_{1,2}}{\partial d_2'}\right] = 0 \quad (P \lesseqgtr 1). \tag{14}$$

Equations (13) and (14) are second-order ordinary differential equations in d_1 and d_2, respectively. The procedure of solving equations (13) and (14) can be considerably simplified by introducing the non-dimensional layer thicknesses d_1^* and d_2^* given by

$$d_{1,2} = d_{1,2}^* \sqrt{(\gamma x/U_\infty)}. \tag{15}$$

The variational principles (11) and (12) are subject to transformation (15) and the resulting Euler–Lagrange equations are obtained as simple polynomial equations

$$\frac{\partial L_{1,2}}{\partial d_1^*} = 0 \tag{16}$$

and

$$\frac{\partial L_{1,2}}{\partial d_2^*} = 0 \quad (P \lesseqgtr 1) \tag{17}$$

the explicit expressions of which are given by equations (A1)–(A3) in the Appendix.

The hydrodynamical boundary layer thickness d_1^* is obtained as the only real and positive root of equation (16). The polynomial equations (17) are solved numerically for given values of P and n and it is found that every pair of P and n values correspond to only one real and positive root d_2^*.

The dimensionless heat transfer coefficient at the wall is calculated with the help of the relation

$$Nu_l = \sqrt{(\gamma x/U_\infty (T_0 - T_\infty)^2)}(J_q/L_\lambda)_{y=0} \tag{18}$$

where the explicit expression for $(J_q/L_\lambda)_{y=0}$ is given by equations (A4) and (A5) in the Appendix.

The local heat transfer in the case of axisymmetric stagnation point flow is given by

$$\overline{Nu_l} = \sqrt{(\gamma X/U_\infty (T_0 - T_\infty)^2)}\left(-\frac{\partial T}{\partial Y}\right)_{Y=0}. \tag{19}$$

Using Mangler's transformation (4) and the relation

$$\left(\frac{\partial}{\partial Y}\right)_{Y=0} = \frac{x}{l}\left(\frac{\partial}{\partial y}\right)_{y=0} \tag{20}$$

we obtain the equation

$$\overline{Nu_l} = \sqrt{3}Nu_l. \tag{21}$$

5. ANALYSIS OF RESULTS

Whenever a problem is treated with a new method, it is customary to compare the obtained results with other numerical results in order to estimate the accuracy involved.

Table 1 exhibits the comparison of Sibulkin's numerical values with the present solutions. It can be easily observed that our approximate analytical solutions are as good as numerical results and the error hardly exceeds 0.5%.

An interesting feature of the present solution is the vanishing of the local heat transfer at $n = -2/3$ for any P.

In order to observe the behaviour of local heat transfer, we plot the values of $\overline{Nu_l}$ against Prandtl number in Figs. 1 and 2. Figure 1 predicts $\overline{Nu_l}$ values in the range $0 < P \leqslant 1$ with the help of five curves corresponding to different values of n. Figure 2 deals with $\overline{Nu_l}$ values in the range $1 \leqslant P < \infty$. Figures 1 and 2 establish the fact that $\overline{Nu_l}$ increases with n and that the increase is quite rapid when n is large. Further, it can be easily seen that $\overline{Nu_l}$ approaches a limiting value when P tends to zero.

6. CONCLUDING REMARKS

This paper presents a complete analytical answer to heat transfer in axisymmetric stagnation point flow. The governing equations of the problem are transformed into simple

Table 1. Comparison of present results with numerical solutions ($n = 0$)

P	Sibulkin [3] (numerical)	Present values (approximate analytical)
0.6	0.625	0.627
0.8	0.700	0.700
1	0.763	0.762
2	0.988	0.985
10	1.760	1.751

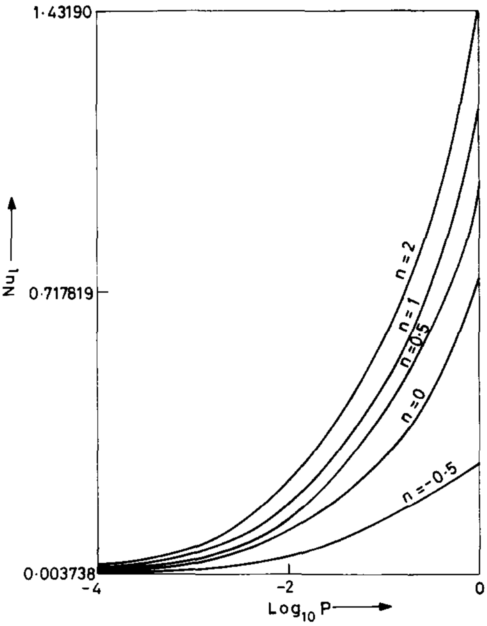

FIG. 1. Local Nusselt number as a function of $\log_{10} P$ ($10^{-4} \leqslant P \leqslant 1$) for various values of n.

polynomial equations (A1)–(A3) the coefficients of which are functions of the independent parameters P and n. These equations offer any practising engineer a rapid way of obtaining heat transfer at the wall for given values of P and n. One should also note that the solution of the present problem is obtained with remarkable ease by this variational procedure especially when compared with the formidable task of solving the governing equations numerically. The amount of labour, time and cost involved in the present approach is certainly less than that of the numerical procedure. The method of solution exhibited here has the further advantage of obtaining analytical solution for the problem rather than simply displaying a table of numerical values. The agreement of the present results with available numerical solutions establishes the fact that the present thermodynamic method, based on sound physical reasoning, is a powerful tool for obtaining a rapid analytical solution to boundary layer problems.

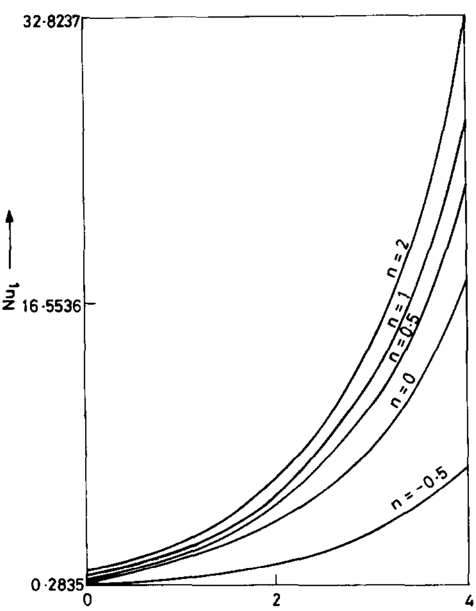

FIG. 2. Local Nusselt number vs $\log_{10} P$ ($1 \leqslant P \leqslant 10^4$) for various values of n.

Acknowledgement—The author would like to thank Prof. I. Gyarmati and Prof. P. Singh for valuable suggestions and encouragement.

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APPENDIX

Substitution of $d_{1,2}$ from equation (15) in the variational principles (11) and (12) and the variation of these principles with respect to the independent parameters d_1^* and d_2^* yield the Euler–Lagrange equations

$$\frac{\partial L_{1,2}}{\partial d_1^*} = 0$$

or

$$d_1^{*4}(21.9994) - d_1^{*2}(65.47603) - 4000 = 0 \qquad \text{(A1)}$$

or

$$d_1^* = 3.879972$$

$$\frac{\partial L_{1,2}}{\partial d_2^*} = 0$$

or

$$\begin{aligned} &d_2^{*9}P^2(3.27381n^2 + 6.547619n + 3.809524) \\ &\quad - d_2^{*7}[P(13.92857n + 17.14285) + P^2 d_1^{*2}(1.875n^2 \\ &\quad + 1.25n)] - d_2^{*5}[60 + Pd_1^{*2}(7.5)n + P^2 d_1^{*4}(5.083333n \\ &\quad - 1.00595n^2)] - d_2^{*4}[Pd_1^{*3}(35 - 7.5n) \\ &\quad + P^2 d_1^{*5}(0.26461n^2 + 0.3924504n + 6.882843)] \\ &\quad + d_2^{*3}[Pd_1^{*4}(1.07145)n + P^2 d_1^{*6}(8.66788n \\ &\quad - 0.213474n^2)] + d_2^{*2}[Pd_1^{*5}(49.28572 - 2.14284n) \\ &\quad + P^2 d_1^{*7}(0.109016n^2 - 5.544807n + 7.006965)] \\ &\quad - Pd_1^{*7}(12.44047 - 0.1785754n) - P^2 d_1^{*9}(0.005337n^2 \\ &\quad - 0.558244n + 2.232084) = 0 \quad (P \leqslant 1) \end{aligned} \qquad \text{(A2)}$$

and

$$\begin{aligned} &d_2^{*10}P^2(1.000874n^2 + 1.501312n \\ &\quad + 0.5900341) - d_2^{*9}d_1^*P^2(9.46042n^2 \\ &\quad + 14.49676n + 5.844686) + d_2^{*8}d_1^{*2}P^2(38.93358n^2 \\ &\quad + 61.52536n + 25.68462) - d_2^{*7}d_1^{*3}P^2(77.94642n^2 \\ &\quad + 126.4583n + 54.82128) + d_2^{*6}d_1^{*4}P^2(67.77597n^2 \\ &\quad + 112.9599n + 51.40695) - d_2^{*5}d_1^{*3}P(64.28574n + 50) \\ &\quad + d_2^{*4}d_1^{*4}P(253.5714n + 209.5239) \\ &\quad - d_2^{*3}d_1^{*5}P(337.5n + 312.5) \\ &\quad - 600d_1^{*6} = 0 \quad (P \geqslant 1). \end{aligned} \qquad \text{(A3)}$$

The value of d_1^* from equation (A1) is substituted in equations (A2) and (A3), respectively, and these equations are solved numerically for d_2^* with given values of P and n. The values of d_1^* and d_2^* which are uniquely determined as above are substituted in the following local energy flux expressions:

$$(-J_q/L_\lambda)_{y=0} = (PU_\infty/\gamma)(T_0-T_\infty)[1.5d'_1(3/20\Delta - 31/420\Delta^3)+d'_2(-3/8+3/40\Delta^2-3/280\Delta^4) + (nd_2/x)(-3/8+0.25/\Delta-3/40\Delta^2 + 1/280\Delta^4)+0.5(d_2/x)(-0.25+0.2/\Delta^2 - 17/210\Delta^4)] \quad (P \leqslant 1) \tag{A4}$$

$$= (PU_\infty/\gamma)(T_0-T_\infty)[1.5d'_1(0.2\Delta^2-\Delta^3/6 + 3\Delta^4/70)+d'_2(-0.6\Delta+3\Delta^2/8-3\Delta^3/35) + (nd_2/x)(-0.3\Delta+\Delta^2/8-3\Delta^3/140) + 0.5(d_2/x)(-0.2\Delta+\Delta^2/12 - \Delta^3/70)] \quad (P \geqslant 1) \tag{A5}$$

and the local heat transfer is computed.

Int. J. Heat Mass Transfer. Vol. 30, No. 11, pp. 2444–2445, 1987
Printed in Great Britain

0017-9310/87 \$3.00+0.00
Pergamon Journals Ltd.

A note on the series solutions of momentum and energy equations for heat transfer from a semi-infinite plate

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(*Received May* 1986)

1. INTRODUCTION

THE LAMINAR flow situation that is our concern here is zero pressure gradient and unity Prandtl number fluid flow over a semi-infinite plate of uniform temperature. For the flow under consideration, Meksyn [1] presented series solutions of the governing momentum and energy equations; the relevant equations and boundary conditions are

$$f'''+ff'' = 0 \tag{1}$$

$$\theta''+f\theta' = 0 \tag{2}$$

$$f(0) = f'(0) = 0; \quad f'(\infty) \to 1 \tag{3}$$

$$\theta(0) = 1; \quad \theta(\infty) \to 0. \tag{4}$$

Here f and θ denote the non-dimensional stream-function and temperature, respectively; primes denote derivatives with respect to y. In brief, Meksyn uses the Blasius series

$$f = \sum_{N=2}^{\infty} A_N y^N/N! = a(y^2/2!)-a^2(y^5/5!) + 11a^3(y^8/8!)-375a^4(y^{11}/11!)+27\,897a^5(y^{14}/14!)\ldots \tag{5}$$

where A_N are the coefficients and $f''(0) = a$, to integrate equations (1) and (2). The end result is that the temperature distribution is a combination of two series: one being in terms of the incomplete gamma function and the other for $\theta'(0)$. The series for $\theta'(0)$ is given by

$$\theta'(0) = -0.478/[1+1/45-1/405\ldots]. \tag{6}$$

For the flow under consideration, the Reynolds analogy suggests that skin-friction is a direct measure of wall heat transfer rate [2]. However, relation (6) does not lead to this explicit relation. It appears that only the functional analysis of the Reynolds analogy between momentum and heat transfer can provide such an explicit relation. Also, it appears (to the author's knowledge) that no attempt was made in the past to arrive at this explicit relation using series solutions of equations (1)–(4). The aim of this note is to show that the series solutions of equations (1)–(4) can be used to develop this explicit relation between the skin-friction and wall heat transfer rate.

2. ANALYSIS

For the sake of convenience, we convert equations (1)–(4) into initial value problems using the following transformations and initial conditions:

$$Y = y/s, \quad F = sf, \quad a = 1/s^3, \quad (d\theta/dy)_{y=0} = c \text{ (say)}. \tag{7}$$

The initial value problems to be solved are

$$F'''+FF'' = 0 \tag{8}$$

$$\theta''+F\theta' = 0 \tag{9}$$

$$F(0) = F'(0) = 0, \quad F''(0) = 1 \tag{10}$$

$$\theta(0) = 1, \quad \theta'(0) = cs = b \text{ (say)}. \tag{11}$$

Here (and in what follows) primes denote derivatives with respect to Y. The parameters s and b are to be estimated satisfying the conditions that $F'(Y) \to s^2$ and $\theta(Y) \to 0$ as $Y \to \infty$.

Using Maclaurin's series expansion one obtains from equations (8)–(11)

$$F = (Y^2/2!)-(Y^5/5!)+11(Y^8/8!)-375(Y^{11}/11!) + 27\,897(Y^{14}/14!)-\cdots+R_M \tag{12}$$

$$\theta = 1+bY-b(Y^4/4!)+11b(Y^7/7!)-375b(Y^{10}/10!) + 27\,897b(Y^{13}/13!)\cdots+R_P. \tag{13}$$

Here R_M and R_P denote the remainders. Comparing series (12) and (5), we note that they differ by the scaling factors introduced in equations (7).

Recently, Torok and Advani [3] have shown that a series solution to a non-linear initial value problem can be obtained via infinitesimal generators. Expressing equations (8) and (10) as three first-order differential equations (for the sake of brevity, details are not given here), one obtains series (12). In this case, this series represents a continuous group of transformations parameterized by Y. Given a point on a trajectory, which is an invariant curve of the transformation group [3], a point is mapped onto another along the trajectory as Y advances. This possibly implies that series (12) is not only true for small Y [2, 4], but for all Y.

However, series (12) does not appear to converge so easily in the sense that F' (obtained from series (12)) does not attain its asymptotic values as $Y \to \infty$. Shank's transformation [5] applied to only five terms of the series appears to accelerate the convergence rate; e.g. from Table 1 (this table contains the results for $Y = 3$ and 6) we see that at $Y = 6$, repeated use of this transformation drastically reduces the F' value from 52 826 ($= S_5$) to 2.07! Noting the observation of Van Dyke [6] that at least 15 terms are required to obtain the